

# Unsupervised Learning: K-Means Clustering

Rafiq Islam

2024-09-28

## Table of contents

Introduction . . . . .	1
Mathematics Behind K-Means . . . . .	2
Objective Function (Inertia) . . . . .	2
How to Choose the Best $k$ Value? . . . . .	3
1. The Elbow Method . . . . .	3
2. Silhouette Score . . . . .	3
3. Gap Statistic . . . . .	3
Python Implementation of K-Means . . . . .	4
Synthetic Data . . . . .	4
Real Data . . . . .	8
K-Means is a powerful and widely used clustering algorithm, but it has limitations, such as assuming spherical clusters of equal sizes. . . . .	14
Limitations of K-Means Clustering . . . . .	14
1. <b>Assumption of Spherical Clusters</b> . . . . .	14
2. <b>Sensitivity to Initialization</b> . . . . .	14
3. <b>Needs to Specify <math>k</math> in Advance</b> . . . . .	15
4. <b>Outliers and Noise Sensitivity</b> . . . . .	15
5. <b>Equal Cluster Size Assumption</b> . . . . .	15
6. <b>Non-Convex Shapes</b> . . . . .	15
References . . . . .	16

## Introduction

Clustering is a fundamental technique in unsupervised learning where the goal is to group similar data points into clusters. One of the most popular algorithms for clustering is **K-Means**. K-Means is a centroid-based algorithm that partitions the dataset into  $k$  clusters. The algorithm iterates over data points, assigning each to one of  $k$  centroids (cluster centers),

and then updates the centroids based on the current assignments. The objective is to minimize the sum of squared distances (also known as inertia) between each data point and its assigned centroid.

## Mathematics Behind K-Means

The K-Means algorithm works through the following key steps:

1. **Initialization:** Randomly select  $k$  points from the dataset as initial centroids.
2. **Assignment Step:** For each data point, assign it to the closest centroid based on the Euclidean distance:

$$\text{distance}(x_i, \mu_j) = \sqrt{\sum_{d=1}^D (x_i^d - \mu_j^d)^2}$$

where:

- $x_i$  is the  $i$ -th data point.
  - $\mu_j$  is the  $j$ -th centroid.
  - $D$  is the number of features (dimensions).
3. **Update Step:** After all data points are assigned, recalculate the centroid of each cluster as the mean of all data points assigned to it:

$$\mu_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_i$$

where  $n_j$  is the number of points in cluster  $j$ .

4. **Repeat:** The assignment and update steps are repeated until the centroids no longer change or the maximum number of iterations is reached.

## Objective Function (Inertia)

The objective of K-Means is to minimize the following cost function, also called inertia or within-cluster sum of squares:

$$J = \sum_{j=1}^k \sum_{i=1}^{n_j} \|x_i - \mu_j\|^2$$

This measures how compact the clusters are, i.e., how close the points within each cluster are to their centroid.

## How to Choose the Best $k$ Value?

One of the critical tasks in K-Means clustering is selecting the optimal number of clusters ( $k$ ). Several methods can be used:

### 1. The Elbow Method

The most common way to determine the best  $k$  is the **elbow method**. It involves plotting the inertia (the sum of squared distances from each point to its assigned cluster centroid) for different values of  $k$ . The point where the inertia starts to flatten out (forming an elbow) is considered a good choice for  $k$ .

### 2. Silhouette Score

The **silhouette score** measures how similar each point is to its own cluster (cohesion) compared to other clusters (separation). It ranges from -1 to 1:

- 1 indicates that the point is well inside its cluster.
- 0 means the point is on the boundary between two clusters.
- Negative values indicate the point may have been assigned to the wrong cluster.

### 3. Gap Statistic

The **gap statistic** compares the total within-cluster variation for different values of  $k$  with the expected value under null reference distribution. The optimal number of clusters is where the gap statistic is the largest.

---

## Python Implementation of K-Means

### Synthetic Data

Let's implement K-Means clustering using Python with visualizations and explore how to choose the best value of  $k$  using the elbow method.

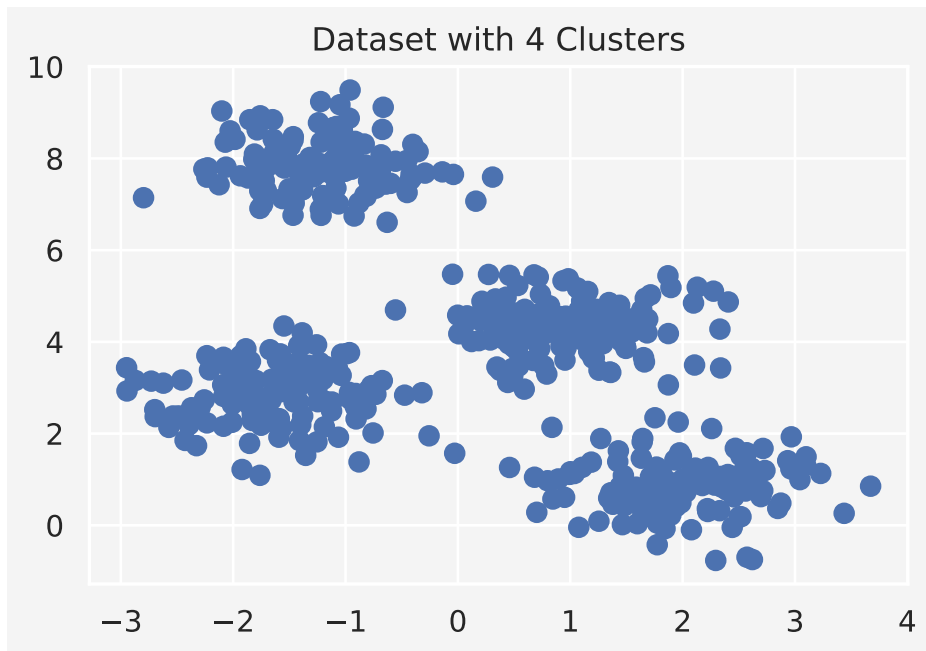
```
import numpy as np
import matplotlib.pyplot as plt
from sklearn.cluster import KMeans
from sklearn.datasets import make_blobs
from sklearn.metrics import silhouette_score

# For plotting purposes
import seaborn as sns
sns.set()
```

We'll create a simple dataset with 4 distinct clusters for visualization.

```
# Create a dataset with 4 clusters
X, y = make_blobs(n_samples=500, centers=4, cluster_std=0.60, random_state=0)

# Visualize the dataset
plt.scatter(X[:, 0], X[:, 1], s=50)
plt.title('Dataset with 4 Clusters')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



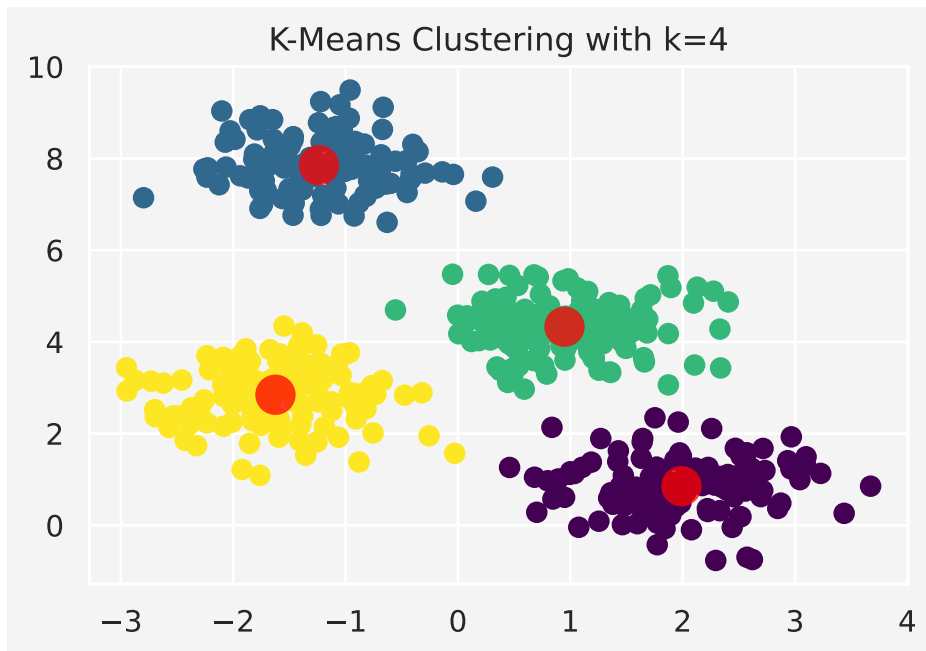
We can now apply K-Means clustering with different values of  $k$  and observe how the clusters are formed.

```
# Fit KMeans with k=4 (since we know we generated 4 clusters)
kmeans = KMeans(n_clusters=4)
kmeans.fit(X)

# Predict clusters
y_kmeans = kmeans.predict(X)

# Plot the clustered data
plt.scatter(X[:, 0], X[:, 1], c=y_kmeans, s=50, cmap='viridis')

# Plot the centroids
centers = kmeans.cluster_centers_
plt.scatter(centers[:, 0], centers[:, 1], c='red', s=200, alpha=0.75)
plt.title('K-Means Clustering with k=4')
plt.savefig('kmeans.png')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```

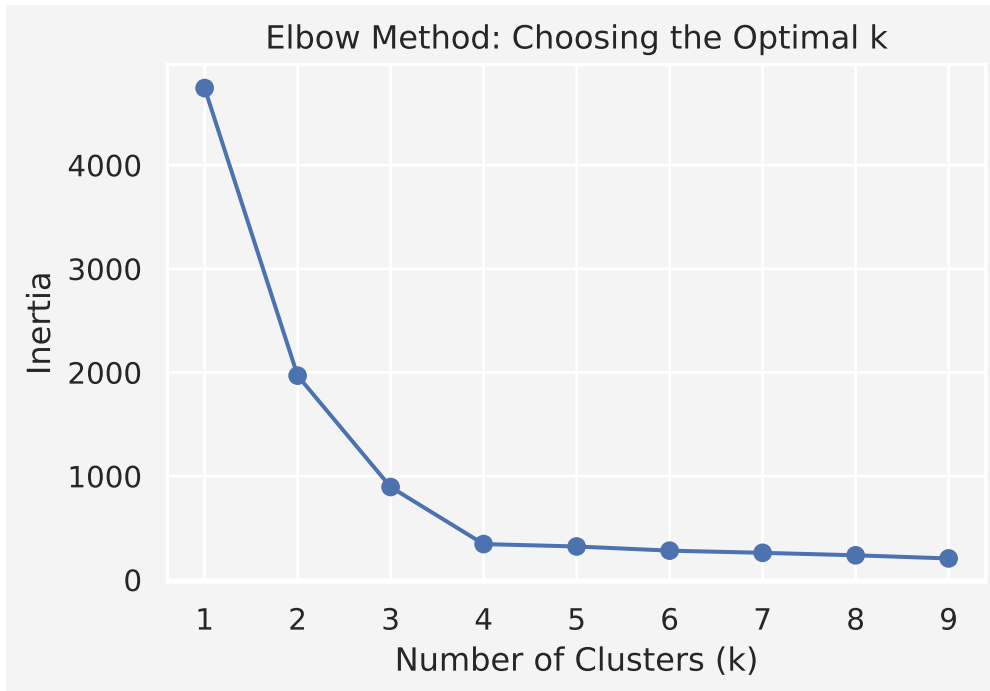


To determine the optimal number of clusters, we'll plot the inertia for different values of  $k$  using the elbow method.

```
# Test multiple k values
inertia = []
k_values = range(1, 10)

for k in k_values:
    kmeans = KMeans(n_clusters=k)
    kmeans.fit(X)
    inertia.append(kmeans.inertia_)

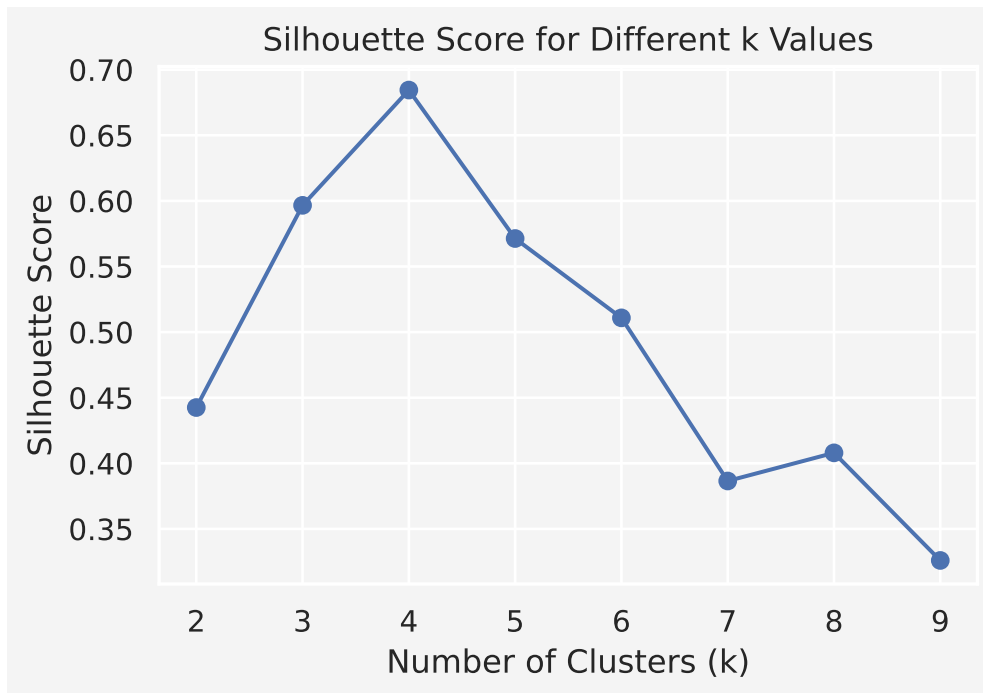
# Plot the inertia vs. k values
plt.plot(k_values, inertia, marker='o')
plt.title('Elbow Method: Choosing the Optimal k')
plt.xlabel('Number of Clusters (k)')
plt.ylabel('Inertia')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



We see that the curve starts to flatten at  $k = 4$ , suggesting this is a good choice for the number of clusters. Let's also compute the silhouette score for different values of  $k$  to confirm our choice.

```
sil_scores = []
for k in range(2, 10):
    kmeans = KMeans(n_clusters=k)
    kmeans.fit(X)
    labels = kmeans.predict(X)
    sil_scores.append(silhouette_score(X, labels))

# Plot Silhouette Score vs. k
plt.plot(range(2, 10), sil_scores, marker='o')
plt.title('Silhouette Score for Different k Values')
plt.xlabel('Number of Clusters (k)')
plt.ylabel('Silhouette Score')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



## Real Data

### Description:

This dataset contains information about customers of a shopping mall, including their annual income, spending score, gender, and age.

**Goal:** Our goal is to segment customers into different groups based on their spending behavior and income.

### Columns:

- **CustomerID:** Unique identifier for each customer.
- **Gender:** The gender of the customer (Male or Female).
- **Age:** Age of the customer.
- **Annual Income:** Annual income of the customer in thousands of dollars.
- **Spending Score:** A score assigned by the mall based on customer behavior and spending patterns.

**Data Source:** You can find the Mall Customer Segmentation data on [Kaggle](#).

```
import pandas as pd
mall = pd.read_csv('Mall_Customers.csv')
mall.head()
```



	CustomerID	Gender	Age	Annual Income (k\$)	Spending Score (1-100)
0	1	Male	19	15	39
1	2	Male	21	15	81
2	3	Female	20	16	6
3	4	Female	23	16	77
4	5	Female	31	17	40

```
# Data Information
print(mall.info())
print('\n')
# Check for Missing Data
print(mall.isnull().sum())
print('\n')
# Data Description
mall.rename(columns={'CustomerID':'ID','Annual Income (k$)':'Income','Spending Score (1-100)':'Spending Score (1-100)'})
cmall = mall.drop('ID',axis=1)
print(cmall.describe().loc[['mean','std','min','max']].T)
```

```
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 200 entries, 0 to 199
Data columns (total 5 columns):
#   Column                Non-Null Count  Dtype
---  -
0   CustomerID            200 non-null   int64
1   Gender                200 non-null   object
2   Age                  200 non-null   int64
3   Annual Income (k$)    200 non-null   int64
4   Spending Score (1-100) 200 non-null   int64
dtypes: int64(4), object(1)
memory usage: 7.9+ KB
None
```

```
CustomerID      0
Gender          0
Age             0
Annual Income (k$) 0
Spending Score (1-100) 0
dtype: int64
```

	mean	std	min	max
Age	38.85	13.969007	18.0	70.0
Income	60.56	26.264721	15.0	137.0
SpendingScore	50.20	25.823522	1.0	99.0

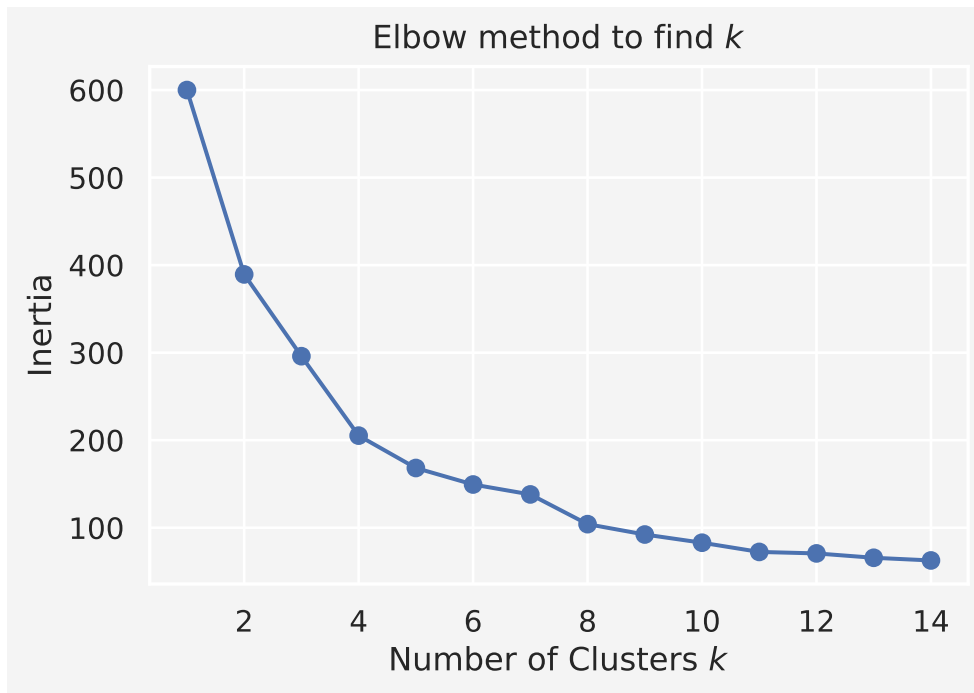
**Pre-Process:** Since our data contains categorical variable **Gender**, we need to encode this column and scale the numerical features like **Age**, **Annual Income**, and **Spending Score**.

```
from sklearn.preprocessing import StandardScaler
X = mall[['Age', 'Income', 'SpendingScore']]
scaler = StandardScaler()
X_scaled = scaler.fit_transform(X)
print(X_scaled[:5])
```

```
[[ -1.42456879 -1.73899919 -0.43480148]
 [ -1.28103541 -1.73899919  1.19570407]
 [ -1.3528021  -1.70082976 -1.71591298]
 [ -1.13750203 -1.70082976  1.04041783]
 [ -0.56336851 -1.66266033 -0.39597992]]
```

Next we use the Elbow method to find the best  $k$ , the number of clusters

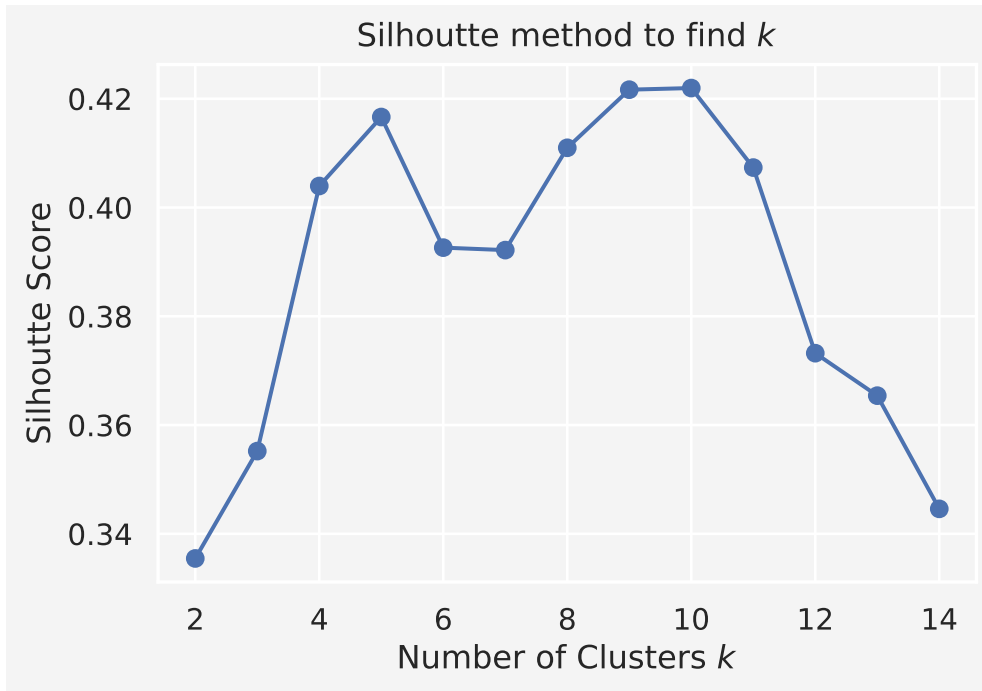
```
k_values = range(1,15)
inertia = []
for k in k_values:
    kmeans = KMeans(n_clusters=k, random_state=123)
    kmeans.fit(X_scaled)
    inertia.append(kmeans.inertia_)
plt.plot(k_values,inertia, marker='o')
plt.title('Elbow method to find $k$')
plt.xlabel('Number of Clusters $k$')
plt.ylabel('Inertia')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



The elbow point in the plot (where the decrease in inertia starts to slow) helps determine the optimal number of clusters. Let's say we find that  $k = 5$  looks like a reasonable choice from the plot.

To further validate the choice of  $k$ , let's compute the **silhouette** score for different cluster numbers. A higher **silhouette** score indicates better-defined clusters

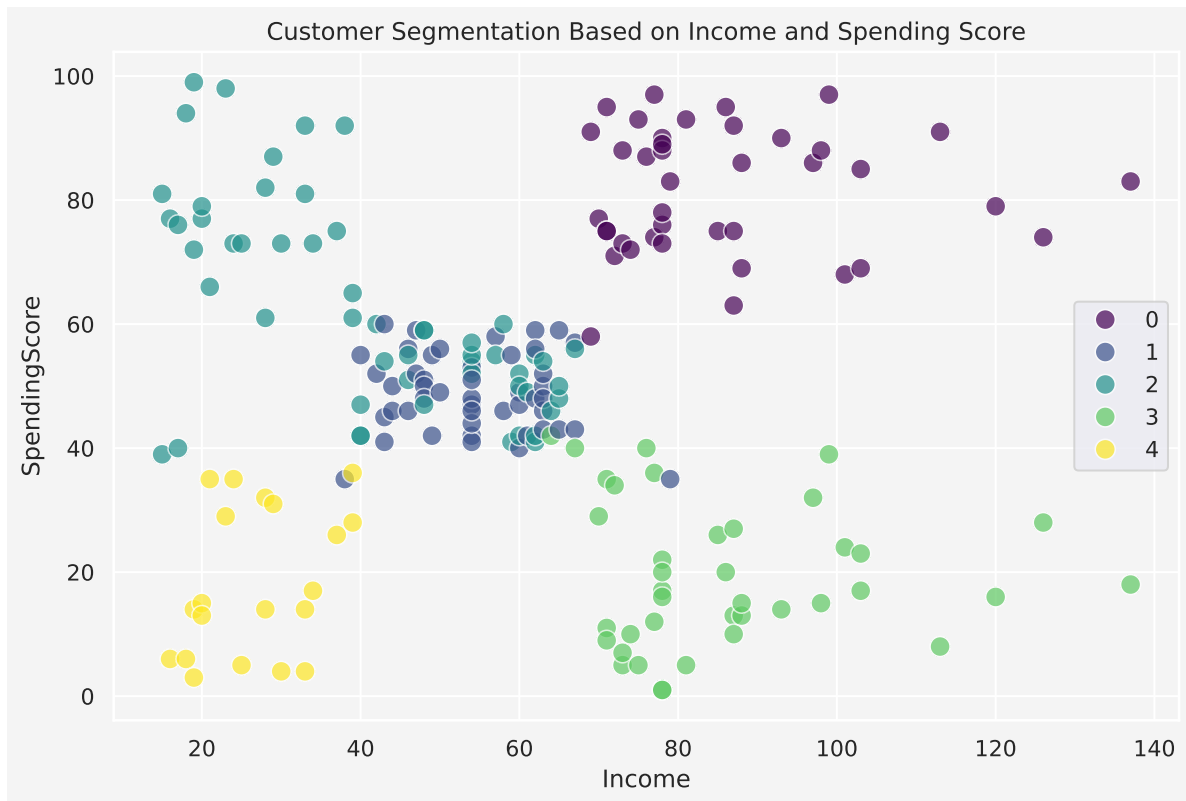
```
sil_scores = []
for k in range(2,15):
    kmeans = KMeans(n_clusters=k, random_state=123)
    labels = kmeans.fit_predict(X_scaled)
    sil_scores.append(silhouette_score(X_scaled,labels))
plt.plot(range(2,15),sil_scores, marker='o')
plt.title('Silhoutte method to find $k$')
plt.xlabel('Number of Clusters $k$')
plt.ylabel('Silhoutte Score')
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```



Next, we apply  $k = 5$  clusters

```
plt.figure(figsize=(9.5,6))
kmeans = KMeans(n_clusters=5, random_state=123)
mall['Cluster'] = kmeans.fit_predict(X_scaled)
print(mall.head())
sns.scatterplot(
    x='Income', y='SpendingScore', hue='Cluster',
    data=mall, palette='viridis', s=100, alpha=0.7
)
plt.title('Customer Segmentation Based on Income and Spending Score')
plt.legend()
plt.gca().set_facecolor('#f4f4f4')
plt.gcf().patch.set_facecolor('#f4f4f4')
plt.show()
```

	ID	Gender	Age	Income	SpendingScore	Cluster
0	1	Male	19	15	39	2
1	2	Male	21	15	81	2
2	3	Female	20	16	6	4
3	4	Female	23	16	77	2
4	5	Female	31	17	40	2



Analyze the segments

```
cluster_summary = mall.drop(columns=['Gender','ID']).groupby('Cluster').mean()
print(cluster_summary)
```

	Age	Income	SpendingScore
Cluster			
0	32.875000	86.100000	81.525000
1	55.638298	54.382979	48.851064
2	25.185185	41.092593	62.240741
3	39.871795	86.102564	19.358974
4	46.250000	26.750000	18.350000

Now say we have two new customers

```
new_customer = {'ID':[201,202], 'Gender':['Male','Female'], 'Age': [30,50], 'Income': [40,70], 'SpendingScore': [60,80]}
new_customer = pd.DataFrame(new_customer)
print(new_customer)
```

	ID	Gender	Age	Income	SpendingScore
0	201	Male	30	40	70
1	202	Female	50	70	20

We would like to know in which cluster they belong.

```
X_new = new_customer[['Age', 'Income', 'SpendingScore']]
X_new_sc = scaler.transform(X_new)
cluster_labels = kmeans.predict(X_new_sc)
print(cluster_labels)
```

```
[2 3]
```

**K-Means is a powerful and widely used clustering algorithm, but it has limitations, such as assuming spherical clusters of equal sizes.**

## Limitations of K-Means Clustering

While K-Means is a widely used clustering algorithm due to its simplicity and scalability, it has several notable limitations:

### 1. Assumption of Spherical Clusters

K-Means assumes that clusters are spherical and have roughly the same size. This assumption may not hold true in real-world datasets, where clusters may have different shapes and densities. For example, if clusters are elongated or irregularly shaped, K-Means may not perform well.

- **Solution:** Use algorithms like **DBSCAN** (Density-Based Spatial Clustering of Applications with Noise) or **Spectral Clustering**, which do not assume any specific shape for the clusters.

### 2. Sensitivity to Initialization

K-Means is sensitive to the initial selection of centroids. Different initializations can lead to different final clusters, and in some cases, the algorithm may converge to suboptimal solutions. To address this, the algorithm is often run multiple times with different initializations (e.g., using the `k-means++` initialization method).

- **Solution:** Use the `k-means++` initialization, which ensures that centroids are chosen in a way that increases the likelihood of converging to an optimal solution.

### 3. Needs to Specify $k$ in Advance

One of the main limitations is that K-Means requires the number of clusters ( $k$ ) to be specified in advance. This can be a challenge when the number of clusters is unknown, and choosing the wrong  $k$  can lead to poor clustering results.

- **Solution:** Use the **Elbow Method**, **Silhouette Score**, or the **Gap Statistic** to estimate the best value for  $k$ .

### 4. Outliers and Noise Sensitivity

K-Means is highly sensitive to outliers, as they can significantly affect the position of centroids. An outlier will either form its own cluster or distort the positions of nearby centroids, leading to incorrect clustering.

- **Solution:** Preprocess your data by removing outliers or use clustering methods like **DBSCAN**, which can handle outliers more effectively by considering them as noise.

### 5. Equal Cluster Size Assumption

The algorithm tends to assign roughly equal-sized clusters because it minimizes variance. This can be a problem if clusters in your data have highly varying sizes. Small clusters might be absorbed into larger ones.

- **Solution:** Use **Hierarchical Clustering**, which can naturally handle different cluster sizes.

### 6. Non-Convex Shapes

K-Means struggles with data where clusters have non-convex shapes, such as two overlapping rings or crescent shapes. It partitions the space into Voronoi cells, which are convex, leading to poor clustering results in non-convex structures.

- **Solution:** Algorithms like **Spectral Clustering** or **Gaussian Mixture Models (GMM)** can better handle non-convex clusters.

---

## References

### 1. K-Means Algorithm:

- MacQueen, J. B. (1967). “Some Methods for Classification and Analysis of Multivariate Observations”. Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Statistics.
- Hartigan, J. A., & Wong, M. A. (1979). “Algorithm AS 136: A K-means clustering algorithm”. Journal of the Royal Statistical Society. Series C (Applied Statistics), 28(1), 100-108.

### 2. Choosing k (Elbow Method & Silhouette Score):

- Rousseeuw, P. J. (1987). “Silhouettes: A graphical aid to the interpretation and validation of cluster analysis”. Journal of Computational and Applied Mathematics, 20, 53-65.

### 3. Inertia and the Elbow Method:

- Tibshirani, R., Walther, G., & Hastie, T. (2001). “Estimating the number of clusters in a dataset via the gap statistic”. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 63(2), 411-423.

---

Share on



You may also like